Unit - II

Equations of Motion of a fluid:

$F = MA$

Pressure at a point in a fluid at Rest:

Pascal's Laws for Static fluid:

i) When the fluid is at rest, the pressure at an internal point P is the Same in all directions, ~~Thus in the above~~ the orientation of δA is immaterial.

ii) The pressure at all points at the Same depth below the free horizontal Surface of a fluid at rest is the Same. At a point P distance h below the free Surface, the pressure is $\pi + \rho g h$, where π is the atmospheric pressure at the free Surface, ρ the fluid density and g the acceleration due to gravity.

iii) If any pressure is applied to the free Surface of a fluid, it is transmitted equally to all parts of the fluid.

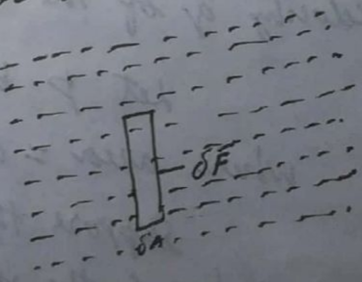
Pressure at a point of a Moving fluid:

Let P be a point in a ideal (inviscid) fluid moving with velocity $\bar{q}$. we insert an elementary rigid plane are δA into this fluid at point P.

This plane area also moves with the velocity $\bar{q}$ of the local fluid at P.

If δF denotes the force exerted on one side of δA by the fluid particle on the other side.

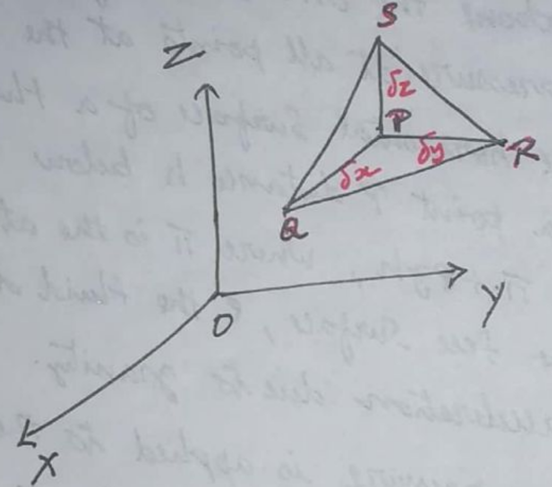
then this forces will act normal to δA.



If we assume that $\lim_{\delta A \to 0} \frac{\delta F}{\delta A}$ exists uniquely, then this limit is called the (hydrodynamics) fluid pressure at point P and is denoted by p.

Theorem: Prove that the pressure p at a point P in a moving inviscid fluid is same in all directions.

Proof:



Let $\bar{q}$ be the velocity of the fluid.

We consider an elementary tetrahedron PQRS of the fluid at a point P of the moving fluid.

Let the edges of the tetrahedron be $PQ = \delta x$, $PR = \delta y$, $PS = \delta z$ at time 't'.

where $\delta x, \delta y, \delta z$ are taken along the co-ordinate axes OX, OY, OZ respectively.

This tetrahedron is also moving with the velocity $\bar{q}$ of the local fluid at P.

Let p be the pressure on the face QRS where area $= \delta S$

Suppose that $\langle l, m, n \rangle$ are the direction cosines of the normal to δS drawn outward from the tetrahedron,

Then,
$$l\delta s = \text{projection of the area } \delta s \text{ on } yz\text{-plane}$$
$$= \text{area of face (triangle) } PRS$$
$$= \frac{1}{2}\delta y\,\delta z = \frac{\delta y\,\delta z}{2}$$

$|||^{ly}$ $$m\delta s = \text{area of face } PQS = \frac{\delta z\,\delta x}{2}$$
$$n\delta s = \text{area of face } PQR = \frac{\delta x\,\delta y}{2}$$

The total force exerted by the fluid, outside the tetrahedron, on the face QRS is
$$= -p\,\delta s\,(l\bar{i} + m\bar{j} + n\bar{k})$$
$$= -p\,(l\delta s\,\bar{i} + m\delta s\,\bar{j} + n\delta s\,\bar{k})$$
$$= \frac{-p}{2}(\delta y\,\delta z\,\bar{i} + \delta z\,\delta x\,\bar{j} + \delta x\,\delta y\,\bar{k}) \longrightarrow (1)$$

Let p_x, p_y, p_z be the pressure on the face PRS, PQS, PRQ.

The forces exerted on these faces by the exterior fluid are
$$\frac{1}{2}p_x\,\delta y\,\delta x\,\vec{i}, \quad \frac{1}{2}p_y\,\delta x\,\delta z\,\vec{j}, \quad \frac{1}{2}p_z\,\delta x\,\delta y\,\vec{k} \longrightarrow (2)$$
respectively.

Thus, the total surface force on the tetrhedron is

$(1)+(2) \Rightarrow$
$$= \frac{-p}{2}(\delta y\,\delta z\,\bar{i} + \delta z\,\delta x\,\bar{j} + \delta x\,\delta y\,\bar{k}) + \frac{1}{2}p_x\,\delta y\,\delta x\,\vec{i}$$
$$+ \frac{1}{2}p_y\,\delta x\,\delta z\,\vec{j} + \frac{1}{2}p_z\,\delta x\,\delta y\,\vec{k}$$

$$= \frac{1}{2}\left[(p_x - p)\,\delta y\,\delta z\,\vec{i} + (p_y - p)\,\delta z\,\delta x\,\vec{j} + (p_z - p)\,\delta x\,\delta y\,\vec{k}\right] \longrightarrow (3)$$

In addition to surface (fluid forces), the fluid may be subjected to body forces which are due to external causes such as gravity.

Let $\vec{F}$ be the mean body force per unit mass within the tetrahedron.

Volume of the tetrahedron $PQRS = \frac{1}{3} h\,\delta S$

i.e., $\frac{1}{6}\delta x\,\delta y\,\delta z$, where h is the perpendicular from P on the face QRS.

Thus, the total force acting on the tetrahedron

$$PQRS = \frac{1}{6}\rho\bar{F}\,\delta x\,\delta y\,\delta z \quad \longrightarrow (4)$$

where, ρ is the mean density of the fluid.

From (3) & (4),

the net force acting on the tetrahedron

$$(1) \leftarrow = \frac{1}{2}\left[(p_x-p)\,\delta y\,\delta z\,\bar{i} + (p_y-p)\,\delta z\,\delta x\,\bar{j} + (p_z-p)\,\delta x\,\delta y\,\bar{k}\right] + \frac{1}{6}\rho\bar{F}\,\delta x\,\delta y\,\delta z$$

Now, the acceleration of the tetrahedron is $\frac{D\bar{q}}{Dt}$ and the mass $\frac{1}{6}\rho\,\delta x\,\delta y\,\delta z$ of fluid inside it is constant.

Thus, the equation of motion of the fluid contained in the tetrahedron =

$$(2) \quad \frac{1}{2}\left[(p_x-p)\,\delta y\,\delta z\,\bar{i} + (p_y-p)\,\delta x\,\delta z\,\bar{j} + (p_z-p)\,\delta x\,\delta y\,\bar{k}\right] + \frac{1}{6}\rho\bar{F}\,\delta x\,\delta y\,\delta z$$

$$= \frac{1}{6}\rho\,\delta x\,\delta y\,\delta z\left(\frac{D\bar{q}}{Dt}\right) \qquad (\because F = m\bar{a})$$

$\leftarrow (2) + (1)$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + \bar{q}\cdot\nabla$$

i.e., ~~(p_x-p)~~

$$2 \Rightarrow \quad (p_x-p)\,l\,\delta s\,\bar{i} + (p_y-p)\,m\,\delta s\,\bar{j} + (p_z-p)\,n\,\delta s\,\bar{k} + \frac{1}{3}\rho\bar{F}h\,\delta s = \frac{1}{3}\rho h\,\delta s\,\frac{D\bar{q}}{Dt}$$

on dividing by δs

and letting the tetrahedron shrink to zero about P, in which case $h \to 0$, it follows that

$$p_x - p = 0 \quad , \quad p_y - p = 0 \quad , \quad p_z - p = 0$$

i.e., $\boxed{p_x = p_y = p_z = p}$ $\longrightarrow$ (5)

$\because$ the choice of axes is arbitrary, it establishes that at any point P of a moving ideal fluid, the pressure p is same in all directions.

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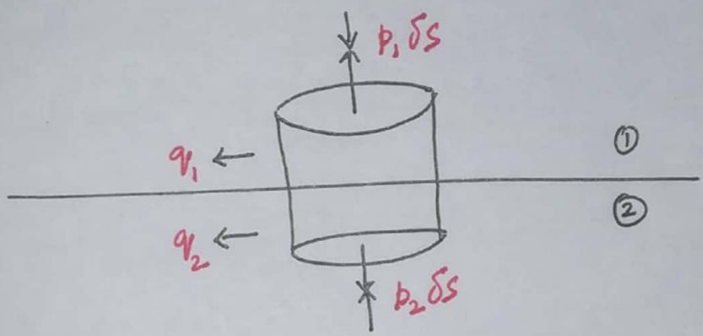
Conditions at a Boundary of Two Inviscid Immiscible Fluids:



It shows two fluids separated by a plane boundary, their velocities at P on the boundary being q_1, q_2 respectively. Consider a small cylindrical hat-box shaped element of normal section δs containing P and projecting into both fluids, its generators being normal to the surface. Since there is no fluid transfer across the boundary,

$$p_1 \delta s = p_2 \delta s$$

$$\boxed{p_1 = p_2}$$

In particular for the case of a liquid in contact with the atmosphere, the pressure at the free surface is the same as that of the atmosphere.

When the boundary is curved, however, the above condition has to be modified to account for the effects of surface tension.

3.4 Euler's Equation of Motion of an ideal Fluid

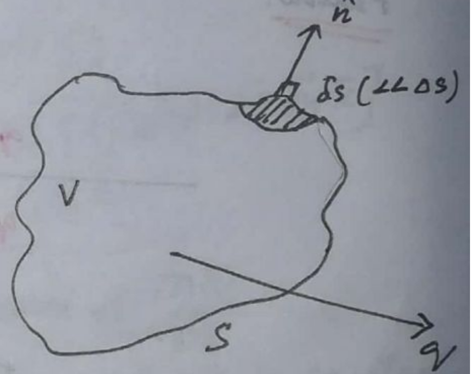
(Equation of Conservation of Momentum)

To obtain Euler's dynamical equation, we shall make use of Newton's Second law of motion.

Consider a region V of fluid bounded by a closed surface S which consists of the same fluid particles at all times.

Let $\bar{q}$ be the velocity and ρ be the density of the fluid

Then ρdV is an element of mass within S and it remain constant.



The linear momentum of volume V is

$$M = \int_V \bar{q}\,\rho\, dV$$

(mass x velocity = momentum)

Rate of change of momentum is

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \bar{q}\,\rho\, dV = \int_V \frac{d\bar{q}}{dt}\,\rho\, dV \longrightarrow (1)$$

The fluid within V is acted upon by two types of forces

The first type of forces are the Surface forces ie simply the pressure p directed along the inward normal at all point of S.

The total Surface force on S is

$$\int_S p(-\hat{n})\,dS = -\int_S p\,\hat{n}\,dS = -\int_V \nabla \cdot p\, dV \longrightarrow (2)$$

(by Gauss divergence theorem

$$\int_S \bar{F}\cdot\hat{n}\,dS = \int_V \nabla\cdot\bar{F}\,dV$$

The Second types of forces are the body forces which are due to some external agent (eg. gravitational). ~~Let~~

Let $\bar{F}$ be the body force per unit mass acting on the fluid.

Then $\bar{F}\ \rho dv$ is the body force on the element of mass ρdv and the total body forces on the mass within V is $\int_V \bar{F}\rho dv \longrightarrow (3)$

F=MA

By Newton's Second law of motion, we have

Rate of change of momentum = Total force

$$\underset{\text{mass}}{M}\ \underset{\text{Acceleration}}{A} = \underset{\text{Force}}{F}$$

$$\Rightarrow \int_V \frac{d\bar{q}}{dt}\cdot \rho dv = \int_V \bar{F}\rho dv - \int_V \nabla p\, dv$$

$$\int_V \left(\frac{d\bar{q}}{dt}\cdot \rho - \bar{F}\rho + \nabla p \right) dv = 0$$

$\because dv$ is arbitrary, we get $\quad (dv \to 0)$

$$\frac{d\bar{q}}{dt}\rho - \bar{F}\rho + \nabla p = 0$$

$$\text{i.e.} \quad \frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p \longrightarrow (4)$$

Equation (4) is known as Euler's Equation of motion.

Other forms

$$\frac{\partial q}{\partial t} + (q\cdot\nabla)q = F - \frac{1}{\rho}\nabla p \qquad \left(\because \frac{d}{dt} = \frac{\partial}{\partial t} + q\cdot\nabla\right.$$

$$\frac{\partial q}{\partial t} + \nabla\left(\tfrac{1}{2}q^2\right) - q\times(\nabla\times q) = F - \frac{1}{\rho}\nabla p.$$

In tensor form, with

x_i as coordinates

u_i as velocity components

F_i as components of body forces.

$(i=1,2,3)$

the equation of motion can be written as

$$\frac{\partial u_i}{\partial t} + u_j u_{i,j} = F_i - \frac{1}{\rho}p_{,i}. \qquad (j=1,2,3)$$

$\Big[\, i=1,\quad \dfrac{\partial u_1}{\partial t} + u_1\dfrac{\partial u_1}{\partial x_1} + u_2\dfrac{\partial u_1}{\partial x_2} + u_3\dfrac{\partial u_1}{\partial x_3} = F_1 - \dfrac{1}{\rho}\cdot\dfrac{\partial p}{\partial x_1}$

$|||^{ly}$ $i=2,\ j=1,2,3$

$i=3,\ j=1,2,3\ \Big]$

$u_1 \to u$ $x_1 \to x$

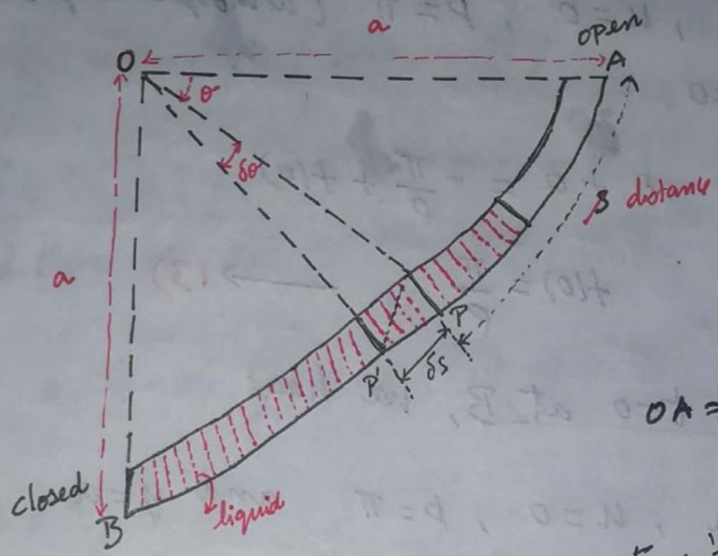
$u_2 \to v$ $x_2 \to y$

$u_3 \to w$ $x_3 \to z$

Example:

1. AB is a tube of small uniform bore forming a quadrantal arc of a circle of radius a and centre O, OA being horizontal and OB vertical with B below O. The tube is full of liquid of density ρ, the end B being closed. If B is suddenly opened, show that initially $\frac{du}{dt} = \frac{2g}{\pi}$, where $u=u(t)$ is the velocity, and that the pressure at a point whose angular distance from A is θ immediately drops to $\rho g a\left(\sin\theta - \frac{2\theta}{\pi}\right)$ above atmospheric pressure. Prove further that when the liquid remaining in the tube subtends an angle β at

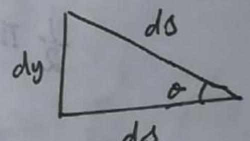
the centre, $\frac{d^2\beta}{dt^2} = \frac{-2g}{a\beta}$



$OA = OB = a = $ radius

Let $AP = s$, arc $AP' = s + \delta s$ at time 't', $PP' = \delta s$

The equation of the element PP' along the arc in the direction of s increasing is



$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + \vec{F}$$

$$= -\frac{1}{\rho}\frac{\partial p}{\partial s} + g\cos\theta \longrightarrow (1) \quad \because \cos\theta = \frac{dy}{ds}$$

$$\left(\because \frac{du}{dt} = \frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial s}\right)$$

$$(1) \Rightarrow \frac{\partial u}{\partial t} + u\cdot\frac{\partial u}{\partial s} = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g\cos\theta$$

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial s}\left(\frac{1}{2}u^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s} + g\frac{dy}{ds}$$

Integrating w.r.t 's', Keeping 't' as constant.

$$\int \frac{\partial u}{\partial t}\,ds + \int \frac{\partial}{\partial s}\left(\frac{1}{2}u^2\right) ds = -\int \frac{1}{\rho}\frac{\partial p}{\partial s}\,ds + \int g\frac{dy}{ds}\,ds$$

$$\frac{\partial u}{\partial t}\cdot s + \frac{1}{2}u^2 = -\frac{1}{\rho}p + gy + f(t) \longrightarrow (2)$$

$\because f(t)$ is a constant and at any time 't'.

$\frac{\partial u}{\partial t}$ is same at all points of the liquid.

Initially $t=0$ at A, we have

$$S=0 \;,\; u=0 \;,\; p=\Pi \;(\text{atmospheric pressure})$$

and $y=0$

$$(2) \Rightarrow \quad 0+0 = -\frac{\Pi}{\rho} + f(0)$$

$$f(0) = \frac{\Pi}{\rho} \longrightarrow (3)$$

Also when $t=0$ at B, we have

$$S=\frac{1}{2}\pi a \;,\; u=0 \;,\; p=\Pi \quad \text{and } y=a$$

$$(2) \Rightarrow \frac{1}{2}\pi a \left.\frac{\partial u}{\partial t}\right|_{t=0} + \frac{1}{2}(0) = \frac{-\Pi}{\rho} + ga + f(0)$$

$$\frac{1}{2}\pi a \left.\frac{\partial u}{\partial t}\right|_{t=0} = \cancel{\frac{-\Pi}{\rho}} + ga + \cancel{\frac{\Pi}{\rho}} \qquad (\text{by } (3))$$

$$\frac{1}{2}\pi \cancel{a} \left.\frac{\partial u}{\partial t}\right|_{t=0} = g\cancel{a}$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \frac{2g}{\pi}$$

hence, $$\left(\frac{du}{dt}\right)_{t=0} = \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial s}\right)_{t=0} = \frac{2g}{\pi}$$

$\because u=0$ when $t=0$

Take $S=a\theta$, $u=0$, $t=0$, $y=a\sin\theta$

$\frac{\partial u}{\partial t} = \frac{2g}{\pi}$ then

$$(2) \Rightarrow \frac{2g}{\pi} a\theta = ga\sin\theta - \frac{p}{\rho} + \frac{\Pi}{\rho} \qquad \text{by}(3)$$

$$\frac{2ga\theta}{\pi} = \rho ga\sin\theta - p + \Pi$$

$$(p-\pi) = \rho g a \sin\theta - \frac{2ga\theta}{\pi}$$

$$(p-\pi) = \rho g a\left(\sin\theta - \frac{2\theta}{\pi}\right)$$

Take $s = a(\pi/2 - \beta)$, $y = a\cos\beta$, $p=\pi$, the condition at the upper force surface. Then

$$(2) \Rightarrow a\left(\frac{\pi}{2}-\beta\right)\frac{\partial u}{\partial t} + \frac{1}{2}u^2 = g a\cos\beta - \frac{\pi}{\rho} + f(t) \longrightarrow (4)$$

Take $s = \frac{a\pi}{2}$, $y=a$, $p=\pi$, the condition at the exit.

Then

$$(2) \Rightarrow \left(a\cdot\frac{\pi}{2}\right)\frac{\partial u}{\partial t} + \frac{1}{2}u^2 = ga - \frac{\pi}{\rho} + f(t) \longrightarrow (5)$$

$$(5)-(4) \Rightarrow a\cdot\frac{\pi}{2}\frac{\partial u}{\partial t} + \cancel{\frac{1}{2}u^2} - a\left(\frac{\pi}{2}-\beta\right)\frac{\partial u}{\partial t} - \cancel{\frac{1}{2}u^2}$$

$$= ga - \cancel{\frac{\pi}{\rho}} + \cancel{f(t)} - ga\cos\beta + \cancel{\frac{\pi}{\rho}} - \cancel{f(t)}$$

$$a\cdot\frac{\pi}{2}\frac{\partial u}{\partial t} - a\left(\frac{\pi}{2}-\beta\right)\frac{\partial u}{\partial t} = ga(1-\cos\beta)$$

$$\cancel{\frac{a\pi}{2}\frac{\partial u}{\partial t}} - \cancel{\frac{a\pi}{2}\frac{\partial u}{\partial t}} + a\beta\frac{\partial u}{\partial t} = ga(1-\cos\beta)$$

$$\cancel{a}\beta\frac{\partial u}{\partial t} = g\cancel{a}(1-\cos\beta)$$

$$\frac{\partial u}{\partial t} = \frac{g}{\beta}\left(2\sin^2\beta/2\right) \longrightarrow (6)$$

Now,

$$u = \frac{\partial s}{\partial t} = \frac{\partial}{\partial t}\left[a\left(\frac{\pi}{2}-\beta\right)\right] = 0 - a\frac{d\beta}{dt} = -a\frac{d\beta}{dt}$$

$$\frac{\partial^2 u}{\partial t^2} = -a\frac{d^2\beta}{dt^2} \longrightarrow (7)$$

from (6), (7) $\Rightarrow$

$$-a\frac{d^2\beta}{dt^2} = \frac{g}{\beta}\left(2\sin^2\beta/2\right) \Rightarrow \boxed{\frac{d^2\beta}{dt^2} = -\frac{2g}{a\beta}\sin^2\left(\frac{\beta}{2}\right)}$$

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3.5 Bernoulli's Equation

Assume that

i) the body forces are conservative

ii) the flow is of the potential kind,

Then there exists the scalar functions Ω, ϕ

Such that $\vec{F} = -\nabla\Omega$, $\bar{q} = -\nabla\phi$

then $\boxed{\nabla \times \bar{q} = 0}$ $(\because \nabla\times\bar{q} = \nabla\times(-\nabla\phi) = 0$

Proof W.K.T the Euler's Equations of motion,

$$\frac{d\bar{q}}{dt} = -\frac{1}{\rho}\nabla p + \vec{F}$$

$$\frac{\partial\bar{q}}{\partial t} + \bar{q}\cdot\nabla\bar{q} = -\frac{1}{\rho}\nabla p + \vec{F}$$

$(\because \nabla\times\bar{q} = 0$

$$\frac{\partial\bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times\underbrace{(\nabla\times\bar{q})}_{0} = -\frac{1}{\rho}\nabla p + \vec{F}$$

$$\frac{\partial\bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\nabla p + \bar{F}$$

$$\frac{\partial}{\partial t}(-\nabla\phi) + \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\nabla p + \bar{F}$$

$$-\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\nabla p + \bar{F}$$

$$-\nabla\left(\frac{\partial\phi}{\partial t}\right) + \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\nabla p - \nabla\Omega \longrightarrow (1)$$

Let $\bar{r}$ be the position vector of the fluid particle at time 't'

let $d\bar{r}$ instantaneous displacement made in the position of particle at this instant 't'.

Then Scalar multiplying the last equation through by dr.

$$-d\bar{r}\ \nabla\left(\frac{\partial \phi}{\partial t}\right) + d\bar{r}\ \nabla\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\ d\bar{r}\ \nabla p - d\bar{r}\ \nabla\Omega$$

Now, using $dr \cdot \nabla\Omega = d\Omega$, we obtain

$$-d\left(\frac{\partial \phi}{\partial t}\right) + d\left(\frac{1}{2}\bar{q}^2\right) = -\frac{1}{\rho}\, d(p) - d\Omega$$

Subject to 't' being constant and integrating on both sides, gives

$$\boxed{-\frac{\partial \phi}{\partial t} + \frac{1}{2}\bar{q}^2 = -\int \frac{dp}{\rho} - \Omega + f(t)} \longrightarrow (2)$$

Eqn. (2) is known as Bernoulli's Equation.

Case (i)

For steady motion (time indep.) constant

$$\frac{\partial \phi}{\partial t} = 0 \quad \text{and} \quad f(t) = \text{constant}$$

$$\frac{1}{2}\bar{q}^2 + \int \frac{dp}{\rho} + \Omega = \text{constant}$$

Case (ii)

The fluid is homogeneous and incompressible

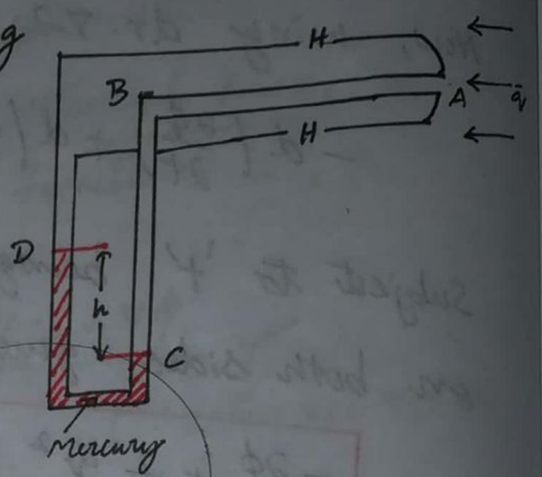
then ρ = constant

$$\frac{1}{2}\bar{q}^2 + \frac{1}{\rho}\int dp + \Omega = \text{constant}$$

$$\frac{1}{2}\bar{q}^2 + \frac{p}{\rho} + \Omega = \text{constant}$$

Examples

The Pitot tube



It is used for measuring fluid velocities.

It is desired to measure the velocity $\bar{q}$ of a stream of water. Then

the inner tube BA is placed upstream of the flow.

The outer tube contains holes such as H.

Let p be the pressure in the stream where the fluid velocity is $\bar{q}$.

Then p is also the pressure on the inside and outside of the hole and

also at the meniscus D of the mercury in the U-tube.

Considering the stream entering the tube AB, this is brought to rest at the meniscus C.

Let p_0 be the pressure. Then by applying Bernoulli's equation along the streamline extending from infinity, entering the inner tube at A and being finally brought to rest at C, we have

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{p_0}{\rho}$$

where ρ is the density of the water.

Hence

$$\frac{1}{2}q^2 = \frac{p_0}{\rho} - \frac{p}{\rho}$$

$$q^2 = \frac{2(p_0 - p)}{\rho}$$

$$q = \sqrt{\frac{2(p_0-p)}{\rho}}$$

The pressure difference $p_0 - p$ is measured from the difference in levels of the mercury.

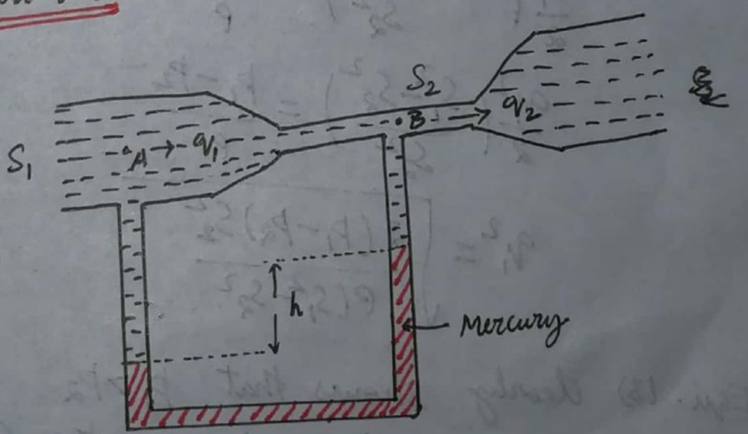
In fact if h is this difference and σ the density of the mercury, then

$$p_0 - p = \sigma g h$$

Thus the ambient velocity of the stream of water is measurable.

* ~~The~~ device is suitable for measuring subsonic airstreams.
* Low speeds air may be treated as an incompressible fluid.

2) <u>Venturi tube</u>



It is a device for measuring flow in a pipe.

The pipe is steadily constricted from a section S_1 to a much smaller section S_2, a U-tube serving as a mercury manometer being inserted between the broad and narrow section at A and B.

Let q_1, q_2 be the fluid velocities at A, B and p_1, p_2 the pressures. The sectional area being S_1, S_2 at these locations, ~~the eqn~~

The equation of continuity, Taking the fluid to be incompressible (density = constant) is

$$q_1 S_1 = q_2 S_2 \longrightarrow (1)$$

Applying Bernoulli's equation along the central streamline from A to B gives

$$\frac{p_1}{\rho} + \frac{1}{2} q_1^2 = \frac{p_2}{\rho} + \frac{1}{2} q_2^2 \longrightarrow (2)$$

ρ being the density of the fluid.

Eliminating q_2 from (1) & (2), we find

$$\frac{p_1}{\rho} + \frac{1}{2} q_1^2 = \frac{p_2}{\rho} + \frac{1}{2} \frac{q_1^2 S_1^2}{S_2^2}$$

$$-\frac{1}{2} q_1^2 + \frac{1}{2} \frac{q_1^2 S_1^2}{S_2} = \frac{p_1}{\rho} - \frac{p_2}{\rho}$$

$$\frac{q_1^2}{2}\left(-1 + \frac{S_1^2}{S_2^2}\right) = \frac{p_1 - p_2}{\rho}$$

$$\frac{q_1^2}{2}\left(\frac{S_1^2 - S_2^2}{S_2^2}\right) = \frac{p_1 - p_2}{\rho}$$

$$q_1^2 = \sqrt{\frac{2(p_1 - p_2) S_2^2}{\rho (S_1^2 - S_2^2}} \longrightarrow (3)$$

Equ. (3) clearly shows that $p_1 > p_2$

i.e, the fluid pressure is minimum at a constriction

This result is rather surprising intuitively.

Certain $q_2 > q_1$.

This result must mean that when a 1-dimensional flow is constricted, the pressure energy of the fluid is mostly converted into Kinetic energy.

If h is the difference of meniscus levels in the mercury manometer and σ the density of mercury, then

$$P_1 - P_2 = \sigma g h \longrightarrow (4)$$

so that (3) & (4) enable q_1 to be found.

The mass of liquid flowing through the unconstricted part of the pipe per unit time is then $\rho q_1 S_1$

—x—

3) A long pipe is of length l and has slowly Tapering Cross-Section. It is inclined at angle α to the horizontal and water flows steadily through it from the upper to the lower end. The section at the upper end has twice the radius of the lower end. At the lower end, the water is delivered at atmosphere pressure. If the pressure at the upper end is twice atmospheric, find the velocity of delivery.

Sol.

Let q_1, q_2 be the entry and exit velocities

$2a, a$ the radii of the entry and exit sections.

Take the horizontal through the centre of the entry section as zero level of potential energy.

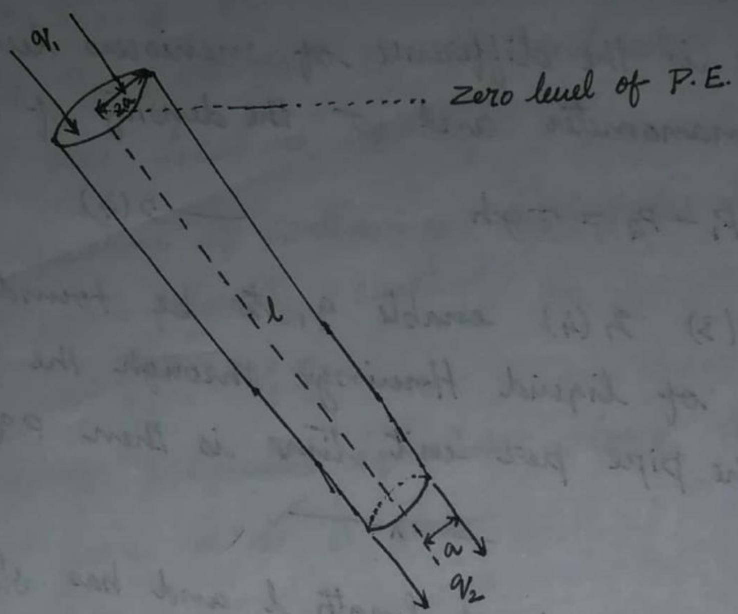
when the flow is steady, taking π as the pressure at exit and 2π that at entry,

Bernoulli's Equation is

$$\frac{P}{\rho} + \frac{1}{2}q^2 + \Omega = \text{Const.}$$

$$\frac{2\pi}{\rho} + \frac{1}{2}q_1^2 + 0 = \frac{\pi}{\rho} + \frac{1}{2}q_2^2 - gl\sin\alpha \longrightarrow (1)$$

$$\frac{\pi}{\rho} + gl\sin\alpha + \frac{1}{2}q_1^2 = \frac{1}{2}q_2^2$$



Since $-gl\sin\alpha$ is the P.E. per unit mass of the gravitational force at the lower end.

Further, the equation of continuity,

$\therefore$ the fluid is incompressible

$$q_1(4\pi a^2) = q_2(\pi a^2) \longrightarrow (2)$$

$$q_1 = \frac{q_2}{4}$$

from (1) & (2) we get

$$\frac{\pi}{\rho} + gl\sin\alpha + \frac{1}{2}\frac{q_2^2}{4^2} = \frac{1}{2}q_2^2$$

$$\frac{q_2^2}{32} - \frac{q_2^2}{2} = -\frac{\pi}{\rho} - gl\sin\alpha \Rightarrow \frac{q_2^2 - 16q_2^2}{32} = \frac{-\pi}{\rho} - gl\sin\alpha$$

$$-15q_2^2 = -32\left(\frac{\pi}{\rho} + gl\sin\alpha\right) \Rightarrow q_2^2 = \frac{32(gl\sin\alpha + \pi/\rho)}{15}$$

$$q_2 = \left[\frac{32(gl\sin\alpha + \pi/\rho)}{15}\right]^{1/2}$$

giving the ~~deriv~~ desired exit velocity.

—x—

3.7 Discussion of the Case of Steady Motion under Conservative Body Forces:

Bernoulli's equation was derived for potential flows under conservative body force.

Let, * The flow is no longer of the Potential kind but is steady

* the body force's are conservative

conditions equations somewhat akin to Bernoulli's equation arise:

Eulers equation of motion may be written as in the form.

$$\frac{\partial q}{\partial t} + (q\cdot\nabla)q = F - \frac{1}{\rho}\nabla p$$

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\times\underbrace{(\nabla\times q)}_{\xi} = F - \frac{1}{\rho}\nabla p$$

where $\xi = \text{curl } q = \nabla\times q$, the vorticity vector.

If the forces are conservative, then $F = -\nabla\Omega$, when the flow is steady.

$$\cancel{\frac{\partial q}{\partial t}}_{0} + \nabla\left(\frac{1}{2}q^2\right) + \nabla\Omega + \frac{1}{\rho}\nabla p = q\times\xi$$

$$\nabla\left(\frac{1}{2}q^2 + \Omega + \int\frac{dp}{\rho}\right) = q\times\xi$$

Scalar multiplying this equation through by dr, a time independent variation in the position vector r, of the fluid particle, gives

$$dr\cdot\nabla\left(\frac{1}{2}q^2 + \Omega + \int\frac{dp}{\rho}\right) = dr\cdot(q\times\xi)$$

$$d\left(\frac{1}{2}q^2 + \Omega + \int\frac{dp}{\rho}\right) = dr\cdot(q\times\xi) \longrightarrow (1)$$

$(\because dr\cdot\nabla\Omega = d\Omega$

Case (i) $q \times \xi = 0$

1) q and ξ are parallel

i.e., when the streamlines and vortex lines coincide. For such motions, q is termed a Beltrami vector.

ii) When $\xi = 0$ $(\because \nabla \times \bar{q} = 0)$ the condition for potential flow

In both case we have

$$d\left(\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right) = 0$$

at all times throughout the entire flow field. Then

$$\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho} = \text{constant} \longrightarrow (2)$$

throughout the entire field of flow.

The constant is the same throughout the entire field since the differential dr. in (1) is any arbitrary small variation of position vector r in the field.

Case (ii) $q \times \xi = 0$ $\quad (q \times (\nabla \times q) = 0)$

Now $q \times \xi$ is perpendicular to the vector q, ξ.

Hence, if $dr \neq 0$, then $dr \cdot (q \times \xi) = 0$

When dr lies in the plane of q, ξ.

Thus if we take the variation dr in the surface containing both the streamlines and vortex lines, then (1) shows that

$$d\left(\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right) = dr \cdot (q \times \xi)$$

$$d\left(\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right) = 0$$

over such a surface, (or) $\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho} = \text{Constant}$ over a surface containing the streamlines and vortex lines.

~~note~~ In eqn. (3), constant is the same everywhere on any one such surface, but that its value varies from one such surface to another. Also (3) holds irrespective of whether the motion is rotational (or) irrotational.